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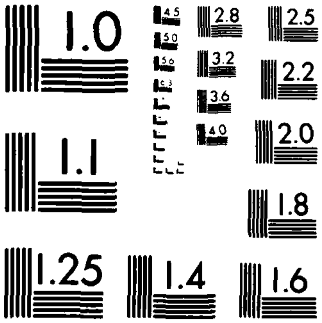

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ON COMPLEX RATIONAL APPROXIMATION BY INTERPOLATION AT PRESELECTED NODES

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ABSTRACT

Let Ω be a finitely connected closed point set in the complex plane with a piecewise smooth boundary $\partial\Omega$. The approximation of functions analytic on Ω by rational functions determined by interpolation or least squares approximation at preselected nodes is discussed. Attention is focussed on simple methods for selecting an appropriate rational space and obtaining a fairly well-conditioned rational basis. Applications include the determination of conformal mappings. Numerical examples illustrate the approximation method.

AMS (MOS) Subject Classifications: 65E05, 65D05, 30C30

Key Words: rational approximation, interpolation, least squares approximation, conformal mapping, analytic continuation.

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SIGNIFICANCE AND EXPLANATION

Complex rational approximation by interpolation has a long history in the theory of approximation. The following numerical questions, however, do not appear to have received much attention:

1) Given a region on which an analytic function shall be approximated by rational functions, and given a set of interpolation points on the boundary of this region, how should one numerically determine a suitable rational space?

2) How does a well-conditioned basis of this rational space look?

3) If one is free to select interpolation points on the boundary, how should they be chosen?

4) Can the selection of the rational space be simplified if one allows least squares approximation instead of interpolation?

The present paper discusses these questions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON COMPLEX RATIONAL APPROXIMATION BY INTERPOLATION AT PRESELECTED NODES

Lothar Reichel

1. Introduction

Let Ω be a closed region of finite connectivity in the complex plane, and assume that the boundary $\partial\Omega$ is piecewise smooth. Let $f(z)$ be a function analytic on Ω and assume that $f(z)$ is explicitly known on $\partial\Omega$ or on a finite point set on $\partial\Omega$. The purpose of the present paper is to describe a numerical method for determining a rational approximant $r(z)$ to $f(z)$ on Ω. The method consists of the following steps

1) select finitely many nodes z_k in the point set of $\partial\Omega$ on which $f(z)$ is known.
2) choose a rational space from which the approximant $r(z)$ is to be selected. The choice will depend on the distribution of nodes z_k.
3) select a well-conditioned basis of the rational space.
4) compute $r(z)$ by interpolation or least squares approximation at the nodes z_k.

Complex approximation by interpolation has a long history in the theory of approximation. Our scheme differs from previously described methods, see [1], [3], [7], [9], in that we use the selection of nodes as starting point. This allows us to treat cases when $f(z)$ is known on a finite point set only, and it also allows us to let the allocation of nodes depend on properties of $f(z)$. In turn we discuss necessary and sufficient conditions on the selection of rational space (section 2), a simple method for choosing rational space (section 3), the selection of nodes (section 4), choice of rational basis (section 5), and approximation on multiply connected regions (section 6). We

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present applications to conformal mapping, and indicate generalizations to the numerical solution of Dirichlet problems for the Laplace equation on multiply connected regions (section 7). Comments on approximation by the discrete least squares method conclude the paper (section 8).

2. Convergence results

Throughout this paper $\{z_{k,m}\}_{k=1}^{m}$ denotes a set of interpolation or least squares nodes on $\partial\Omega$. The set $\{w_{k,n}\}_{k=1}^{n-1}$ denotes a set of not necessarily distinct poles in Ω_c, the complement of Ω with respect to the extended complex plane, and defines the rational space

$$(2.1) \qquad Q_n := \operatorname{span}\{1,(z-w_{1,n})^{-1}, (z-w_{1,n})^{-1}(z-w_{2,n})^{-1},\ldots, \prod_{k=1}^{n-1} (z-w_{k,n})^{-1}\} \quad .$$

The approximation error we measure in the maximum norm

$$\|f\|_{\partial\Omega} := \max_{z\in\partial\Omega} |f(z)| \quad .$$

The next definitions follow [9].

Definitions.

Let the real valued function σ be positive a.e. on a Jordan arc γ, and assume $\int_\gamma \sigma(z)|dz| = 1$. The direction of integration defines an orientation on γ, and we let z^* be the first point of γ. The mapping $F : \gamma \to [0,1]$, $F(\zeta) := \int_{\zeta^*}^{\zeta} \sigma(\zeta)|d\zeta|$ defined by integration along in the positive direction has an inverse a.e. . For a sequence of sets $\{\zeta_{kn}\}_{k=1}^{n}$, $n = 1,2,3\ldots$ of points ζ_{kn} on γ, let for constants $0 < d_1 < d_2 < 1$, $N_n(d_1,d_2)$ denote the number of points of $\{\zeta_{kn}\}_{k=1}^{n}$ on the subarc of γ with end points $F^{-1}(d_1)$ and $F^{-1}(d_2)$. If $\lim_{n\to\infty} \frac{1}{n} N_n(d_1,d_2) = d_2 - d_1$, $\forall d_1,d_2$, $0 < d_1 < d_2 < 1$, then the point sets are said to be uniformly distributed on γ with respect to σ as $n \to \infty$. A set $\{\zeta_{jn}\}_{j=1}^{n}$ of points ζ_{jn} on γ is said to be equidistributed with respect to γ if $F^{-1}(\zeta_{k,n}) - F^{-1}(\zeta_{k-1,n}) = \frac{1}{n}$, $k = 2(1)n$. Sequences of equidistributed sets are uniformly distributed. The definitions carry over to Jordan curves if we let ζ^* be a point on the curve, and identify the curve with an arc with ζ^* both as first and last point.

■

The following theorem covers approximation on bounded simply connected regions. Extensions are provided in the remark below.

Theorem 2.1

Let Γ_{nodes} and Γ_{poles} be piecewise smooth Jordan curves, Γ_{poles} containing Γ_{nodes} in its interior, and $\Gamma_{\text{poles}} \cap \Gamma_{\text{nodes}} = \emptyset$. Let S denote the open region between Γ_{nodes} and Γ_{poles}. Let $U(z)$ solve the Dirichlet problem

$$(2.2)\qquad \begin{cases} U(z) & \text{is harmonic in } S \text{ as a function of } x,y,\ z = x+iy,\ x,y \text{ real} \\ U(z) & \text{is continuous on } S \cup \Gamma_{\text{poles}} \cup \Gamma_{\text{nodes}} \\ U(z) = 1 & \text{on } \Gamma_{\text{nodes}} \\ U(z) = 0 & \text{on } \Gamma_{\text{poles}} \quad . \end{cases}$$

Let $\frac{\partial}{\partial n}$ denote the outward normal derivative with respect to S. Then

$$(2.3)\qquad c := \int_{\Gamma_{\text{nodes}}} \frac{\partial U(z)}{\partial n}\,|dz| = -\int_{\Gamma_{\text{poles}}} \frac{\partial U(z)}{\partial n}\,|dz| > 0 \quad .$$

Let $f(z)$ be a function analytic on and interior to $\Gamma_{\mu_1} := \{z,\ U(z) = \mu_1\}$, where μ_1 is a constant $0 < \mu_1 < 1$. Let $\{z_{k,n}\}_{k=1}^{n}$, $n = 1,2,3,\ldots$ be a sequence of points sets on Γ_{nodes} uniformly distributed with respect to $c^{-1}\frac{\partial U}{\partial n}$ as $n \to \infty$, and let $\{w_{k,n}\}_{k=1}^{n-1}$, $n = 2,3,4,\ldots$ be a sequence of point sets on Γ_{poles} uniformly distributed with respect to $-c^{-1}\frac{\partial U}{\partial n}$. Let the sets $\{w_{k,n}\}_{k=1}^{n-1}$ define a sequence of rational spaces Q_n, $n = 2,3,4\ldots$, see (2.1). Then $r_n \in Q_n$, uniquely determined by interpolating $f(z)$ at points in the set $\{z_{k,n}\}_{k=1}^{n}$, converges to $f(z)$ on and interior to Γ_{μ_2}, for all $0 < \mu_2 < \mu_1$, as $n \to \infty$.

The rate of convergence is given by

$$(2.3)\qquad \overline{\lim_{n\to\infty}} \max_{z\in\Gamma_{\mu_2}} |f(z) - r_n(z)|^{1/n} \le e^{-2\pi(\mu_1-\mu_2)} \quad .$$

If the sequences $\{z_{k,n}\}_{k=1}^{n}$, $\{w_{k,m}\}_{k=1}^{n-1}$ are uniformly distributed with respect to another density function such that

(2.4a) $$V_n(z) := \frac{1}{n}\sum_{k=1}^{n} \ln|z-z_{k,n}| - \frac{1}{n}\sum_{k=1}^{n-1} \ln|z-w_{k,n-1}|$$

(2.4b) $$\lim_{n\to\infty} V_n(z) =: V(z) \text{ is nonconstant on } \Gamma_{\text{nodes}} ,$$

then there is a function $g(z)$ analytic on and interior to Γ_{nodes}, such that $r_n(z) \in Q_n$, $r_n(z_{k,n}) = f(z_{k,n})$, $k = 1(1)n$, and $\|g-r_n\|_{\Gamma_{\text{nodes}}} \to 0$ as $n \to \infty$.

<u>Proof</u>. Equations (2.2) have a unique solution $\hat{U}(z)$ with $\frac{\partial \hat{U}}{\partial n} > 0$ a.e. on Γ_{nodes} and $\frac{\partial \hat{U}}{\partial n} < 0$ a.e. on Γ_{poles}. Green's formula yields (2.3). By studying potentials (2.4a) Walsh has established the connection between the level curves of $\hat{U}$ and the rate of convergence, see the proof of Theorem 9 in Walsh [10], ch. 8. If the limit potential $V(z)$ in (2.4) is nonconstant on Γ_{nodes}, then there are points $\zeta_1 \in \Gamma_{\text{nodes}}$, $\zeta_2 \in$ exterior of Γ_{nodes} such that $V(\zeta_1) > V(\zeta_2)$. Let $g(z) := (z-\zeta_2)^{-1}$, and let $r_n \in Q_n$ interpolate $g(z)$ at $z = z_{k,n}$, $k = 1(1)n$. Then by [10], ch. 8,

$$g(\zeta_1) - r_n(\zeta_1) = \prod_{k=1}^{n} \frac{\zeta_1 - z_{nk}}{\zeta_2 - z_{nk}} \cdot \prod_{k=1}^{n-1} \frac{\zeta_2 - w_{nk}}{\zeta_1 - w_{nk}}$$

and

$$\ln|g(\zeta_1) - r_n(\zeta_1)| = n(V_n(\zeta_1) - V_n(\zeta_2)) .$$

Since $$V_n(\zeta_1) - V_n(\zeta_2) \to V(\zeta_1) - V(\zeta_2) =: \delta > 0,\ n \to \infty ,$$

$$|g(\zeta_1) - r_n(\zeta_1)| = e^{n\delta} \to \infty,\ n \to \infty ,$$

which shows divergence and completes the proof. ∎

Remark 2.1

The distribution of nodes and poles described in the theorem is invariant under conformal mapping, see [10], section 9.12: Let ϕ map S conformally and 1 to 1 onto $\phi(S)$ and be continuous and 1 to 1 on $S \quad \Gamma_{nodes} \quad \Gamma_{poles}$. If the node sets $\{z_{kn}\}_{k=1}^{n}$ are uniformly distributed on Γ_{nodes} with respect to $c^{-1}\frac{\partial U}{\partial n}$ as $n \to \infty$, where U solves (2.2), then the node sets $\{\phi(z_{kn})\}_{k=1}^{n}$ are uniformly distributed on $\phi(\Gamma_{nodes})$ with respect to $c_{\phi}^{-1}\frac{\partial U_{\phi}}{\partial n}$ as $n \to \infty$, where U_{ϕ} solves the Dirichlet problem analogous to (2.2) on the mapped region $\phi(S \cup \Gamma_{nodes} \cup \Gamma_{poles})$, and $c_{\phi} = \int_{\phi(\Gamma_{nodes})} \frac{\partial U_{\phi}}{\partial n}(z)|dz|$. Similarly, if $\{w_{kn}\}_{k=1}^{n-1}$ are uniformly distributed on Γ_{poles} with respect to $-c^{-1}\frac{\partial U}{\partial n}$ as $n \to \infty$, then the sets $\{\phi(w_{kn})\}_{k=1}^{n-1}$ are uniformly distributed on $\phi(\Gamma_{poles})$ with respect to $-c_{\phi}^{-1}\frac{\partial U_{\phi}}{\partial n}$ as $n \to \infty$. Especially theorem 2.1 holds also if Γ_{nodes} is exterior to Γ_{poles}, or if Γ_{nodes} is a piecewise smooth Jordan arc.

■

Remark 2.2

The configuration of curves in theorem 2.1, may consist of several mutually exterior curve pairs $\{\Gamma_{nodes}, \Gamma_{poles}\}$, and the allocation of nodes and poles on each pair can be made independent of the allocation on the other pairs. This follows from the fact that for each curve pair a solution of (2.2) if extended to the exterior of the curve pair would be constant there. Its normal derivative on any other curve would vanish. This remark follows again from [10], ch. 8, theorem 9 and its proof, which covers a more general situation than theorem 2.1 above.

■

We close this section by indicating how theorem 2.1 can be used for computing analytic continuations. Let $f(z)$ be known on a curve Γ_{nodes}. In many physical problems one may know that $f(z)$ is analytic in a specific simply connected region B containing Γ_{nodes} in its interior. Then let Γ_{poles} be he boundary curve of B. The rational approximants r_n computed as described in the theorem converge to $f(z)$ in the interior of Γ_{poles} as $n \to \infty$. The next section discusses how the nodes z_{kn} and poles w_{kn} can be allocated without explicitly solving the Dirichlet problem (2.2).

3. Selection of rational space

We discuss approximation of analytic functions $f(z)$ on simply connected regions. Our starting point is the assumption that a density function σ for the interpolation nodes $\{z_{k,n}\}_{k=1}^{n}$ on Γ_{nodes} is known, and that the nodes are equidistributed w.r.t. σ. We assume $\sigma > 0$ a.e. on Γ_{nodes}. If a set of nodes $\{z_{k,n}\}_{k=1}^{n}$ is given on Γ_{nodes}, then we construct a piecewise linear density function such that the nodes are equidistributed w.r.t. the constructed density function, which we also denote by σ. A set of poles $\{w_{k,n}\}_{k=1}^{n-1}$ defining the space Q_n are obtained by solving (2.2) as an initial value problem: U and $\frac{\partial U}{\partial n} = \sigma$ are known on Γ_{nodes}, and we want to determine other level curves of U, on one of which we allocate the poles $w_{k,n}$.

An initial value problem

Assume that Γ_{nodes} is a smooth Jordan curve. If Γ_{nodes} has corners we round them for the present computations. If Γ_{nodes} is an arc, we replace the arc by a smooth circumscribing curve, or we could proceed as illustrated in section 8. First determine a set of $n - 1$ points $\{\zeta_{k,n}\}_{k=1}^{n-1}$ equidistributed w.r.t. σ. Let W denote the conjugate harmonic function to U such that $W(\zeta_{1,n}) = 0$. The conformal mapping

$$z \to \hat{z} = \phi(z) := \exp(U(z) + iW(z))$$

maps S, see theorem 2.1 for a definition, conformally on an annulus with $\phi(\zeta_{k,n}) = \exp(1 + 2\pi i \frac{k-1}{n})$, $k = 1(1)n-1$. Now assume that S is exterior to Γ_{nodes}. By the conformal invariance noted in remark 2.1, the poles $\{w_{k,n}\}_{k=1}^{n-1}$ should be allocated so that the points $\phi(w_{k,n})$, $k = 1(1)n-1$, are equidistant on a circle concentric with the unit circle. Let ϕ^{-1} be the inverse of ϕ, and let $\hat{z} = e^{t+is}$, $s,t \in \mathbb{R}$. Then

$$z = \phi^{-1}(e^{t+is})$$

(3.1)

$$\zeta_{k,n} = \phi^{-1}\left(e^{1+2\pi i \frac{k-1}{n}}\right) \quad , \quad k = 1(1)n-1 \quad .$$

For a fixed $t = t_0$, the curve $z = z(t_0,s)$, $0 \leq s < 2\pi$, is a level curve of U. We determine such level curves by solving an initial value problem for the Cauchy-Riemann equations for $z = z(t,s)$,

$$\frac{\partial z}{\partial t} = -i \frac{\partial z}{\partial s} \quad . \tag{3.3a}$$

Initial value problems for (3.3a) are ill-posed, but a low accuracy solution suffices for our purpose, and generally we integrate few steps only. The ill-posedness has not caused any difficulty in the present application. On Γ_{nodes} (3.1) yields, with $s_k = 2\pi \frac{k-1}{n}$, $k = 1(1)n$,

$$\frac{\partial z}{\partial s}(1,s_k) \approx \frac{\zeta_{k+1,n} - \zeta_{k-1,n}}{2\Delta s} \quad . \tag{3.4}$$

Substituting (3.4) into (3.3a), integrating in the positive t-direction with Eulers method with $\Delta t = \Delta s$, and denoting the computed approximation of $z(1 + \Delta t, s_k)$ by $w_{k,n}$, yields the scheme

$$w_{k,n} := \zeta_{k,n} - \frac{i}{2}(\zeta_{k+1,n} - \zeta_{k-1,n}),\ k = 1(1)n-1,\ \zeta_n := \zeta_1,\ \zeta_0 := \zeta_{n-1} \quad . \tag{3.5a}$$

The $w_{k,n}$ lie on an approximate level curve of U, and $w_{k,n}$ lies approximately on the same stream line as $\zeta_{k,n}$. Hence, the $w_{k,n}$ are approximately equidistributed w.r.t. $\frac{\partial U}{\partial n}$ on a level curve of U.

Ex. 3.1. Let

$$\Gamma_{nodes} := \{z(t) := x(t) + iy(t),\ x(t) := 1.75 \cdot \cos(t)$$

$$+ 2.625 \cdot \cos(2t) - 2.625,\ y(t) := 3.0625 \cdot \sin(t-0.2) \tag{3.6}$$

$$+ 1.225 \cdot \sin(2t) - 0.6125 \cdot \sin(4t) + 0.875,\ 0 \leq t < 2\pi\} \quad .$$

Allocate 32 points $\zeta_{k,33}$ equidistantly w.r.t. arc length on Γ_{nodes}, and integrate according to (3.5a). Figure 3.1 is obtained.

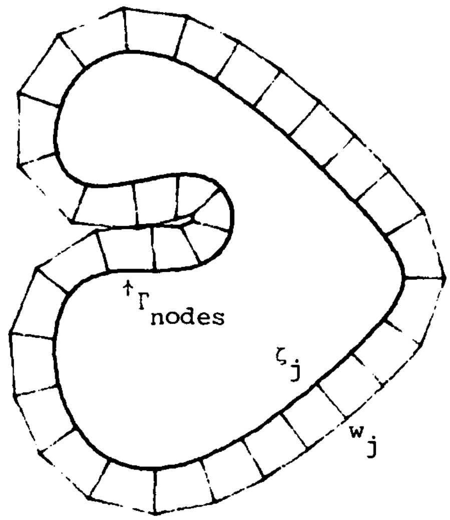

Figure 3.1

The curve (3.6) has been used by Meiss-Markowitz [5] in a quite different example.

■

The integration (3.5a) can be repeated by first letting $\zeta_{k,n} := w_{k,n}$, $k = 1(1)n-1$, and then performing (3.5a). The integration should be repeated until the computed level curve intersects itself, and the $w_{k,n}$ should be allocated on the last non-intersecting or near-non-intersecting computed level curve. We motivate this by considering the case where Γ_{nodes} is the unit circle. Analogous results can be established for more general curves Γ_{nodes}.

Ex. 3.2. Let $\Gamma_{nodes} = \{z : |z| = 1\}$, and let the interpolation nodes be equidistant on Γ_{nodes}. S is exterior to Γ_{nodes}, and level curves of U

are circles $|z| = r > 1$. The poles $w_{k,n}$ will lie equidistantly on a circle $|z| = r_0 > 1$, and

$$U(z) = 1 - \ln|z| \cdot (\ln|z_0|)^{-1} .$$

At any fixed point $\tilde{z}$, $1 < |\tilde{z}| < r_0$, we have that $U(\tilde{z})$ increases with $|z_0|$. Hence, approximation of analytic functions on the unit disk by interpolation at the roots of unity is by theorem 2.1, best done by the family of rational functions which correspond to $r_0 = \infty$, i.e. polynomials.

■

When approximation of functions on the region exterior to Γ_{nodes}, is considered, then S is in the interior of Γ_{nodes}, and (3.3a) is replaced by

(3.3b) $$\frac{\partial z}{\partial t} = i \frac{\partial z}{\partial s} .$$

The corresponding difference equation is

(3.5b) $$w_{k,n} := \zeta_{k,n} + \frac{i}{2} (\zeta_{k+1,n} - \zeta_{k-1,n}), \ k = 1(1)n-1, \ \zeta_n := \zeta_1, \ \zeta_0 := \zeta_{n-1} .$$

Method (3.5) as well as other integration methods for (3.3) have been studied in [6] for the case when the function to be continued analytically is analytic in a simply connected region. The analysis carries without difficulty over to the present situation.

We conclude this section with some computed examples. All computations in this paper were carried out on a VAX/780 in double precision arithmetic, i.e. with 12 significant digits.

Ex. 3.3. Approximate $f(z) := \sqrt{z-\alpha}$ on and interior to the curve Γ_{nodes} of example 3.1., with $\alpha := -2.6325 + i\,1.425$, see figure 3.2

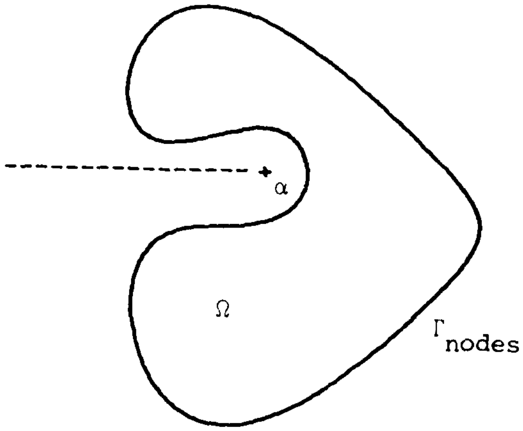

Figure 3.2

The branch of the square root is chosen to make $f(z)$ single valued and analytic in the plane cut along $z := \alpha + t,\ t \leq 0$. For $n = 1 + 32\ell$, $\ell = 1(1)4$, we allocate n interplation nodes $z_{k,n}$ equidistantly w.r.t. arc length on Γ_{nodes}. Q_{32} is defined by the 32 poles $w_{j,33}$ on figure 3.1. $Q_{33+32\ell}$, $\ell = 1,2,3$ are defined by letting $w_{j,33+32\ell} := w_{(j \bmod 32)+1,33}\ \forall_j$. Let $r_n(z)$ denote the element in Q_n such that $r_n(z_{k,n}) = f(z_{k,n})$, $k = 1(1)n$. In figure 3.3 the computed errors are marked with dots.

We note, in passing, that the nodes of course do not have to be allocated exactly equidistantly w.r.t. arc lengths. Nodes could be marked sufficiently accurately with a light pen, and also equations (3.3a,b) are sufficiently simple to allow an approximate graphic solution.

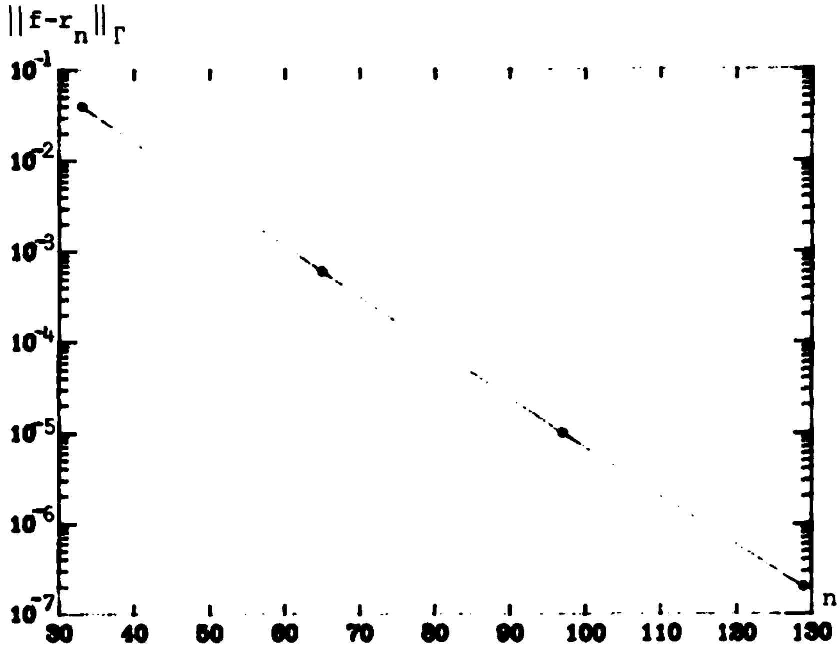

Figure 3.3

Ex. 3.4. In theory, it is also possible to approximate the function $f(z)$ of example 3.3 by polynomials. Let $\{z^F_{k,n}\}^n_{k=1}$ denote a set of n Fejér points for Γ_{nodes}. For their definition, see [3] or [9]. Figure 3.4 shows the points $\{z^F_{k,120}\}^{120}_{k=1}$ marked with crosses on Γ_{nodes}. Interpolation of $f(z)$ in n points defines a polynomial $p_n(z)$ of degree $< n$, and the polynomial sequence $\{p_n(z)\}^\infty_1$ converges maximally to $f(z)$, i.e. $p_n(z)$ converges exponentially to $f(z)$, $n \to \infty$, and there is no polynomial sequence with a higher exponential rate of convergence to $f(z)$.

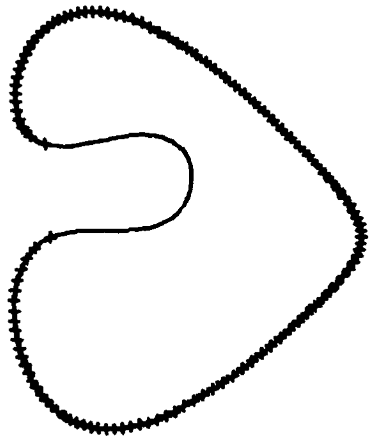

Figure 3.4

Computation of p_n for some n gave the following table

n	$\|f-p_n\|_{\Gamma_{nodes}}$
40	0.60
80	0.42
120	0.42
160	impossible to evaluate.

A slow rate of convergence is combined with difficulties of accurately evaluating $p_n(z)$ for large n. A Lagrange polynomial basis was used, and the Fejér points $z^F_{k,n}$ were determined with 4 significant digits.

■

Ex. 3.5. Approximate $f(z) := \frac{1}{z}\sqrt{(z-z_1)(z-z_2)}$ on and exterior to Γ_{nodes} defined by (3.6), where $z_1 := -4-2i$, $z_2 = i$, see figure 3.5.

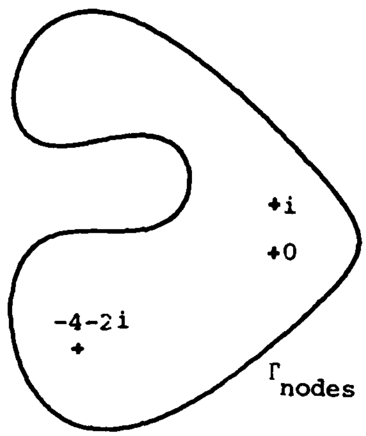

Figure 3.5

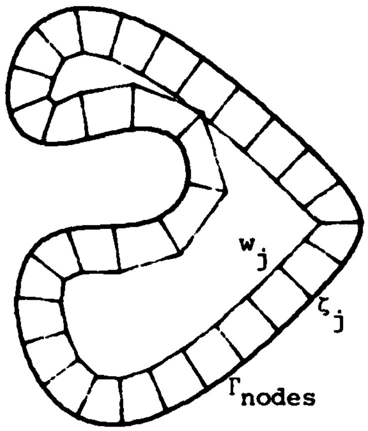

Figure 3.6

The branch of the square root is chosen so that $zf(z)$ is analytic in the finite plane cut between z_1 and z_2. We wish to approximate $f(z)$ by interpolation in 33 nodes equidistant w.r.t. arc length. Figure 3.6 shows 32 points $\zeta_{j,33}$ allocated equidistantly w.r.t. arc length on Γ_{nodes}, and also 32 poles $w_{j,33}$ obtained by applying (3.5b) once. r_{33} denotes the rational interpolant to $f(z)$ in Q_{33}, see below.

Allocate 65 interpolation nodes on Γ_{nodes} equidistantly w.r.t. arc length. Let $w_{j,65} := w_{j+32,65} := w_{j,33}$, $j = 1(1)32$. This defines Q_{65}. Let r_{65} be the interpolant to $f(z)$ in Q_{65}. We obtain

n	$\|f-r_n\|_{\Gamma_{nodes}}$
33	$1\cdot 10^{-2}$
65	$1\cdot 10^{-4}$.

4. Selection of nodes

If $f(z)$, the function to approximate, is known on a discrete point set only, there may be no choice to make. In this section we assume that $f(z)$ is known everywhere on Γ_{nodes}. If no knowledge of the location of the singularities of $f(z)$ is available, we want an allocation of nodes such that the orthogonal distance from a point of Γ_{nodes} to level curves $\{z : u(z)) = \rho\}$, $\rho > 0$, is approximately constant for all points of Γ_{nodes}. Then the rate of convergence will depend on the distance from Γ_{nodes} to a singularity of $f(z)$ closest to Γ_{nodes}. This is, to a first approximation, achieved by allocating the nodes equidistantly w.r.t. arc length.

Knowledge about the location of the singularities of $f(z)$ can be used by allocating more nodes on parts of Γ_{nodes} close to a singularity. The level curves of $U(z)$ will be close to Γ_{nodes} where the node density is highest. Example 7.2 provides an illustration.

5. Rational basis

The basis implicit in the definition (2.1) of Q_n is generally ill-conditioned. If the nodes $z_{k,n}$ and poles $w_{k,n}$ are near-equidistributed with respect to $|c^{-1}\frac{\partial u}{\partial u}|$ on Γ_{nodes} and Γ_{poles}, respectively, then the basis $\ell_0(z) = 1$, $\ell_j(z) := \prod_{\substack{k=1\\k<j}}^{n-1} \frac{z-z_{k,n}}{z_{j,n}-z_{k,n}} \cdot \prod_{k=1}^{n-1} \frac{z_{k,n}-w_{k,n}}{z-w_{k,n}}$, $j = 1(1)n-1$, is fairly well-conditioned. A condition number of a basis we define following Gautschi [ ], with the map $F_n : \mathbb{R}^n \to Q_n : \underline{a} \to \sum_{k=0}^{n-1} a_k \ell_k(z)$, where $\underline{a} = (a_0, a_1, \ldots, a_{n-1})$. Equip $\mathbb{R}^n$ with the maximum norm $\|\underline{a}\|_\infty := \max|a_i|$ and Q_n with the norm $\| \ \|_{\Gamma_{nodes}}$. The induced operator norms are

$$\|F_n\| := \max_{\|\underline{a}\|_\infty = 1} \|\sum_0^{n-1} a_k \ell_k(z)\|_{\Gamma_{nodes}} = \max_{z \in \Gamma_{nodes}} \sum_{k=0}^{n-1} |\ell_k(z)| ,$$

$$\|F_n^{-1}\| := \max_{\|\sum_0^{n-1} a_k \ell_k(z)\|_{\Gamma_{nodes}} = 1} \|\underline{a}\|_\infty < 2 ,$$

and the condition number of the basis $\ell_j(z)$ is

$$\text{cond } F_n := \|F_n\| \, \|F_n^{-1}\| < 2 \max_{z \in \Gamma_{nodes}} \sum_{k=0}^{n-1} |\ell_k(z)| < 2n \cdot \max_{0<k<n-1} \|\ell_k(z)\|_{\Gamma_{nodes}} .$$

Assuming that the z_k, w_k are uniformly distributed with respect to $\ell^{-1}|\frac{\partial U}{\partial n}|$, we have for $k \neq 0$,

$$\ln|\ell_k(z)| = (n-1)\Big(\frac{1}{n-1}\Big(\sum_{\substack{j=1\\j\neq k}}^{n-1} \ln|z-z_{j,n}| - \ln|z_{k,n}-z_{j,n}|\Big)$$

$$- \frac{1}{n-1}\Big(\sum_{j=1}^{n-1} \ln|z-w_{j,n-1}| - \ln|z_{k,n}-w_{j,n-1}|\Big)\Big)$$

$$= (n-1)\,\Big(\int_{\Gamma nodes} \ln|z-\zeta|\,|c^{-1}|\frac{\partial U}{\partial n}(\zeta)|\ |d\zeta| - \int_{\Gamma nodes} \ln|z_{k,n}-\zeta|\,|c^{-1}|\frac{\partial U}{\partial n}(\zeta)|\ |d\zeta| +$$

$$- \int_{\Gamma_{poles}} \ln|z-\zeta|c^{-1}|\frac{\partial U}{\partial n}(\zeta)|\ |d\zeta| + \int_{\Gamma_{poles}} \ln|z_{k,n}-\zeta|c^{-1}|\frac{\partial U}{\partial n}(\zeta)|\ |dz| +$$

$$O(\frac{\ln(n)}{n}))$$

for any fixed $z \in \Gamma_{nodes}$, $z \neq z_{j,n_1}$ as $n \to \infty$. Hence, $|\ell_k(z)| = O(n)$, $k = 0(1)n-1$, for any fixed $z \in \Gamma_{nodes}$, $n \to \infty$. This shows that the basis $\{\ell_j\}_{j=0}^{n-1}$ is reasonably well-conditioned.

6. Multiply connected regions

We consider the approximation of a function $f(z)$ on an exterior doubly connected region. Generalization to more complicated regions are immediate. Let $f(z)$ be analytic on and exterior to the piecewise smooth curves γ_1 and γ_2. γ_i bounds the region Ω_i, see figure 6.1.

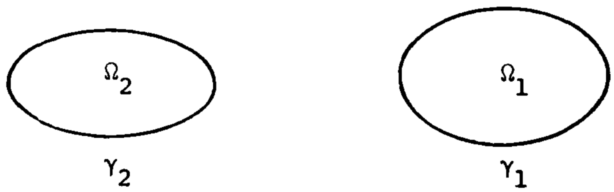

Figure 6.1

We wish to approximate $f(z)$ by a rational function which interpolates $f(z)$ in nodes on γ_1 and γ_2. During the allocation of poles we can regard the problem as if it were composed of the two simpler subproblems: Approximate $f_i(z)$, analytic on and exterior to γ_i, $i = 1,2$, and $f_2(\infty) = 0$. Consider the case $i = 1$. Assume n nodes $\{z_k^{(1)}\}_{k=1}^{n}$ and a density function $\sigma^{(1)}$ are known on γ_1. The method of section 3, yields poles $w_k^{(1)}$, $k = 1(1)n-1$, in Ω_1, and this defines the space

$$(6.1) \qquad Q_n^{(1)} := \operatorname{span}\{1, (z-w_1^{(1)})^{-1}, \ldots, \prod_{k=1}^{n-1} (z-w_k^{(1)})^{-1}\}$$

and basis

$$(6.2) \qquad \begin{cases} \ell_k^{(1)}(z) := \prod\limits_{\substack{j=1 \\ j\neq k}}^{n-1} \dfrac{z-z_j^{(1)}}{z_k^{(1)}-w_j^{(1)}} \prod\limits_{j=1}^{n-1} \dfrac{z_k^{(1)}-w_j^{(1)}}{z-w_j^{(1)}}, & k = 1(1)n-1 \\ \ell_0^{(1)}(z) := 1 \ . \end{cases}$$

Repeat for problem $i = 2$. Nodes $z_k^{(2)}$, $k = 1...m$ are assumed to be known on γ_2. Since $f_2(\infty) = 0$, the rational space we are to construct does not have to contain constants. We can identify the nodes $z_k^{(2)}$ with the points ζ_k of section 3 and obtain poles $w_k^{(2)}$, $k = 1,...,m$, defining the space

$$(6.3) \qquad Q_m^{(2)} := \operatorname{span}\{(z-w_1^{(2)})^{-1},\dots,\prod_{k=1}^{m}(z-w_k^{(2)})^{-1}\}$$

and basis

$$\ell_k^{(2)}(z) := \prod_{\substack{j=1\\ j\neq k}}^{m} \frac{z-z_j^{(2)}}{z_k^{(2)}-z_k^{(2)}} \prod_{j=1}^{m} \frac{z_k^{(2)}-w_j^{(2)}}{z-w_j^{(2)}}, \quad j = 1(1)m \quad .$$

To solve the original problems, we select the function $r_{n,m} \in Q_n^{(1)} \oplus Q_m^{(2)}$ which interpolates $f(z)$ in $\{z_k^{(1)}\}_{k=1}^{n} \cup \{z_k^{(2)}\}_{k=1}^{m}$. Convergence results analogous to theorem 2.1 can be shown, see remark 2.2, provided that both $n,m \to \infty$. The basis $\{\ell_0^{(1)},\dots,\ell_{n-1}^{(1)}, \ell_1^{(2)},\dots,\ell_m^{(2)}\}$ is fairly well-conditioned on $\gamma_1 \cup \gamma_2$, under the assumption that $\|\ell_k^{(1)}\|_{\gamma_2}$, $k = 1(1)n-1$, and $\|\ell_k^{(2)}\|_{\gamma_1}$, $k = 1(1)m$ are small. The method of proof is similar to that used in section 5. The assumption is reasonable due to the relation between nodes and poles.

Ex. 6.1. Let $f(z) = \sqrt{(z+\frac{5}{2})(z+\frac{3}{2})} \cdot (z+2)^{-1} + \sqrt{(z-2-\frac{i}{2})(z-2+\frac{i}{2})} \cdot (z-2)^{-1}$ where the branches are selected so that the first term is analytic in the complex plane cut along the line segment between $z = -\frac{5}{3}$ and $z = -\frac{3}{2}$, and the second term is analytic in the plane cut along the line segment between $z = 2 + \frac{i}{2}$ and $z = 2 - \frac{i}{2}$. Approximate $f(z)$ in the region exterior to both curves

$$\gamma_1 = \{z = 2 + \cos(t) + i\cdot\sin(t)(2+\cos(t))^2,\ 0 \le t < 2\pi\}$$

$$\gamma_2 = \{z = -2 - \sin(t)(2+\cos(t))^2 + i\cdot\cos(t),\ 0 \le t < 2\pi\} \quad .$$

Figure 6.1 shows γ_1, γ_2 and the branch points of $f(z)$ marked with crosses inside γ_1 and γ_2.

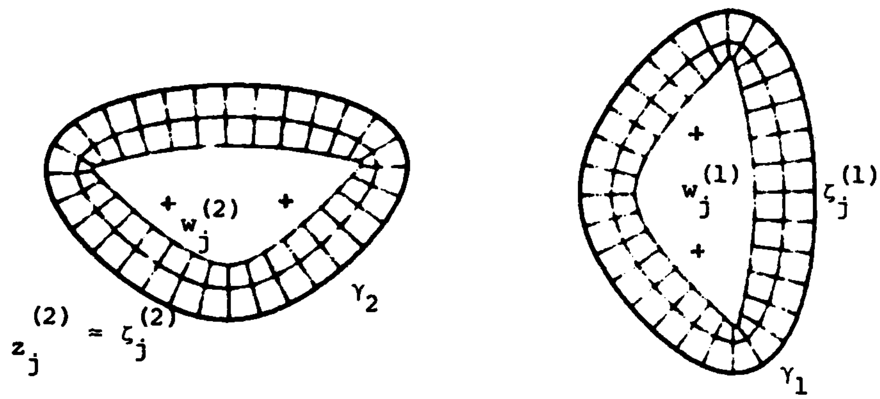

Figure 6.2

We wish to interpolate $f(z)$ in 31 nodes $z_k^{(1)}$ equidistant w.r.t. arc length on γ_1, and in 30 nodes $z_k^{(2)}$ equidistant w.r.t. arc length on γ_2. Figure 6.2 shows 30 points $\zeta_k^{(1)}$ equidistant w.r.t. arc length on γ_1, the 30 points $z_k^{(2)} = \zeta_k^{(2)}$ on γ_2, and the poles $w_j^{(1)}$ and $w_j^{(2)}$ obtained by applying (3.5b) twice for each curve γ_i. Denote the interpolating rational approximant by $r(z)$. Then

$$\|f - r\|_{\gamma_1} = 4 \cdot 10^{-7} \quad , \quad \|f - r\|_{\gamma_2} = 4 \cdot 10^{-7} \quad .$$

■

7. Applications to Dirichlet problems and conformal mapping

The close connection between rational approximation and approximation by rational harmonics, see [11], suggests applications to the numerical solution of Dirichlet problems for the Laplace equation on simply and multiply connected regions. We discuss in some detail the special Dirichlet problems whose solution yields a conformal mapping from a simply connected region to $|w| > 1$ or to $|w| < 1$.

Let Ω be simply connected with boundary $\partial\Omega = \Gamma_{nodes}$ and complement Ω_c. Allocate nodes z_j, $j = 1(1)2n-1$, on Γ_{nodes} for some n. Determine $n-1$ points ζ_j on Γ_{nodes}, such that the distribution functions for z_j and ζ_j agree. Allocate $n-1$ poles w_j in Ω_c by application of (3.5a) or (3.5b). This defines the $2n-1$ - dimensional space of harmonic functions

$$\text{(7.1)} \quad \mathrm{span}\{\ell_0(z),\ \mathrm{Re}(\ell_1(z)),\ \mathrm{Im}(\ell_1(z)),\ldots,\mathrm{Re}(\ell_{n-1}(z)),\mathrm{Im}(\ell_{n-1}(z))\} \quad ,$$

where

$$\text{(7.2)} \quad \begin{cases} \ell_0(z) := 1 \\ \ell_j(z) := \prod\limits_{\substack{k=1 \\ k\neq j}}^{n-1} \dfrac{z-z_{2k}}{z_{2j}-z_{2k}} \prod\limits_{k=1}^{n-1} \dfrac{z_{2k}-w_k}{z-w_t} \quad , \quad j = 1(1)n-1 \quad . \end{cases}$$

An approximate solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = q & \text{on } \partial\Omega \end{cases}$$

is obtained by solving

$$\text{(7.3)} \qquad \mathrm{Re}\Big(\sum_{k=0}^{n-1} a_k \ell_k(z_j)\Big) = g(z_j) \quad , \quad j = 1(1)2n-1$$

for the a_k. This system may not always be uniquely solvable, which is why, in the computed examples below, we solved (7.3) by singular value

decomposition of the matrix. In none of the computed examples the matrix was near-singular.

The special choice

$$g(z) = -\ln|z| \tag{7.4}$$

leads to approximate conformal mappings for Ω provided that $0 \notin \partial\Omega$.

Ex. 7.1. Let Ω be the bounded region of figure 3.5. Compute a conformal mapping $\phi : \Omega \to |w| < 1$, such that $\phi(0) = 0$. Allocate 129 nodes z_j on $\partial\Omega$ equidistantly w.r.t. arc length. Use the poles $\{w_j\}_{j=1}^{32}$ shown in figure 3.1, and define $w_{j+32} := \bar{w}_j$, $j = 1(1)32$. Thus we obtain a basis (7.2) for $n = 65$, and solve (7.3) with $g(z)$ defined by (7.4). This yields

$$\phi_{65}(z) := z \exp\left(\sum_{k=0}^{64} a_k \ell_k(z)\right)$$

which approximates ϕ. Figure 7.1 shows $\phi_{65}(\partial\Omega)$ and a reference circle of radius 1.1. The error $\| |\phi_{65}(z)| - 1 \|_{\partial\Omega} = 7 \cdot 10^{-5}$ is well below the resolution of the picture.

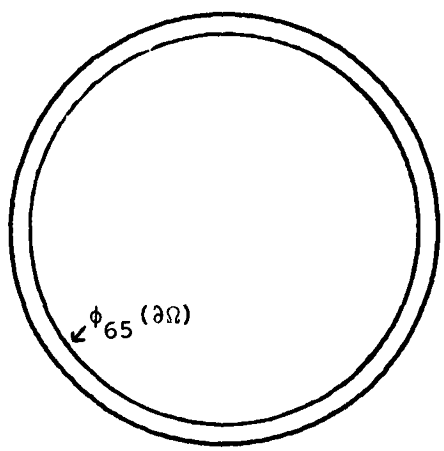

Figure 7.1

Ex. 7.2. Let Ω be the exterior of the curve in figure 7.2. Compute a conformal mapping $\psi : \Omega \to |w| > 1$, such that $\psi(\infty) = \infty$. Allocate 65 nodes z_j on $\partial\Omega$, equidistantly w.r.t. arc length, and use the 32 poles w_j of figure 3.6. Solving (7.3), with g defined by (7.4), one obtains the approximate map

$$\psi_n(z) := z \exp\left(\sum_{k=0}^{n-1} a_k \ell_k(z)\right) \tag{7.5}$$

for $n = 33$.

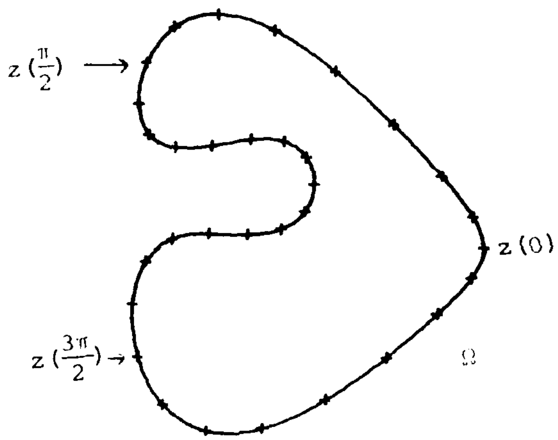

Figure 7.2

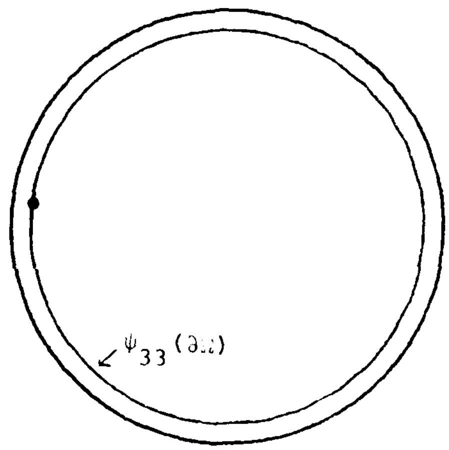

Figure 7.3

Figure 7.3 shows $\psi_{33}(\partial\Omega)$ and a reference circle of radius 1.1. $\| |\psi_{33}(z)| - 1 \|_{\partial\Omega} = 3 \cdot 10^{-2}$. $\psi_{33}(\partial\Omega)$ intersects itself at the blob, which is roughly the image of the part of $\partial\Omega$, which lies strictly interior to the convex hull of $\partial\Omega$. This suggests that in order to achieve higher accuracy more nodes should be allocated between the points $z(\frac{\pi}{2})$ and $z(\frac{3\pi}{2})$ in figure

7.1. The allocation of nodes to be described next is simplified by the fact that the parameter t in (3.6) satisfies $\frac{d(\text{arc lenth})}{\partial t} \approx$ constant, $0 \leq t < 2\pi$. (Figure 2.2 shows 32 points on $\partial\Omega$ equidistant w.r.t. t.) This simplifies the determination of the nodes and poles to be used, but is not essential for the discussion, why we chose not to use this fact in the first part of this example. Figue 7.1 shows 32 points marked with crosses on Γ and allocated equidistantly with respect to the boundary parameter t, see (3.6). In figure 7.4 we have allocated 11 points ζ_j equidistantly with respect to t for $0 \leq t < \frac{\pi}{2}$, 44 points ζ_j equi-distantly with respect to t for $\frac{\pi}{2} \leq t < \frac{3\pi}{2}$, and 11 points equidistantly with respect to t, $\frac{3\pi}{2} \leq t < 2\pi$. By (3.5b), we obtain 66 poles w_j, and finally we allocate 133 nodes on Γ having the same distribution as the ζ_j. This yields the mapping $\phi_{67}(z)$. Figure 7.5 shows $\phi_{67}(\Gamma)$ and a circumscribed concentric reference circle. $\| |\phi_{67}(z)| - 1 \|_\Gamma = 3 \cdot 10^{-3}$.

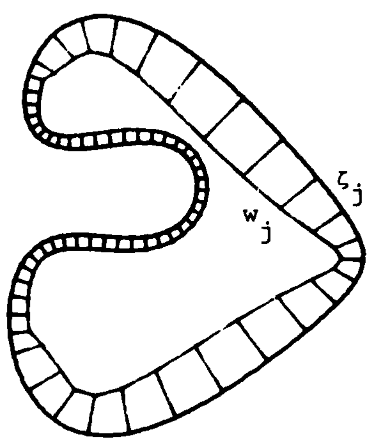

Figure 7.4

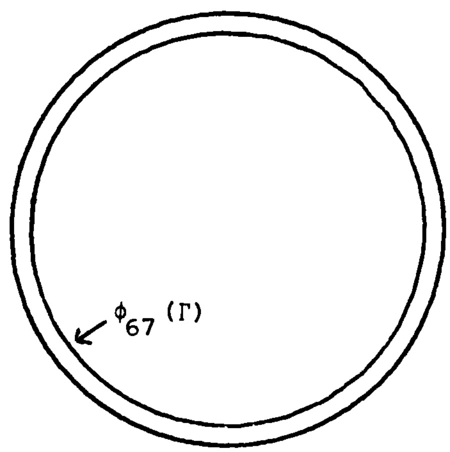

Figure 7.5

The abbreviation error is near the resolution of the plotter. Double the number of poles in figure 7.4 by $w_{j+66} := w_j$, $j = 1(1)66$. This yields $4 \times 66 + 1 = 265$ harmonic basis function, which we determine by interpolation at 265 points z_j, which we allocated with the same density function as we used for ϕ_{67}. This gives $\phi_{133}(z)$ and $\| |\phi_{133}(z)| - 1 \|_\Gamma = 6 \cdot 10^{-6}$. Figure 7.6 shows $\phi_{133}(\partial\Omega)$ and a reference circle of radius 1.1.

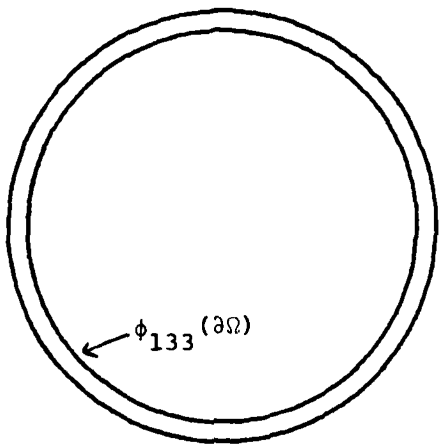

Figure 7.6

8. Least squares approximation

We consider an example, where the distribution of nodes and poles does not agree very well with the conditions of theorem 2.1, and point out that in such cases least squares approximation may give higher accuracy than interpolation.

Approximate $f(z) := \sqrt{z^2-1}$ on $\partial\Omega = [-5i,5i]$, by using function values at equidistant nodes $z_{k,m} := i(10\,\frac{k-1}{m} - 5)$, $k = 1(1)m$, on $\partial\Omega$. The branch of the square root is chosen to make $f(z)$ analytic in the finite plane cut on the real axis from 1 to ∞ and from -1 to $-\infty$. The poles we allocate in a simple manner: for n even, let for some $s > 0$

$$\left\{\begin{array}{l} w_{2k,n} := i(10\,\frac{2k-2}{n-2} - 5) + s \;, \\ w_{2k-1,n} := i(10\,\frac{2k-2}{n-2} - 5) - s \;, \end{array}\right. \qquad k = 1(1)\frac{n}{2} \;.$$

First, consider the selection of s. For s large the distribution of nodes $z_{k,m}$ and poles $w_{k,n}$ does not agree at all with the distribution suggested in theorem 2.1. In fact, for $s = \infty$, i.e. polynomial approximation, approximation by interpolation diverges. This follows from the similarities of our approximation problem and the classical example of Runge, see [2]. On the other hand, for $s > 0$ small the rate of convergence becomes unnecessarily slow.

Ex. 8.1. Let $n = m + 1$ and compute the rational approximant $r_n(z)$ to $f(z)$ for $s = 1,2,3$.

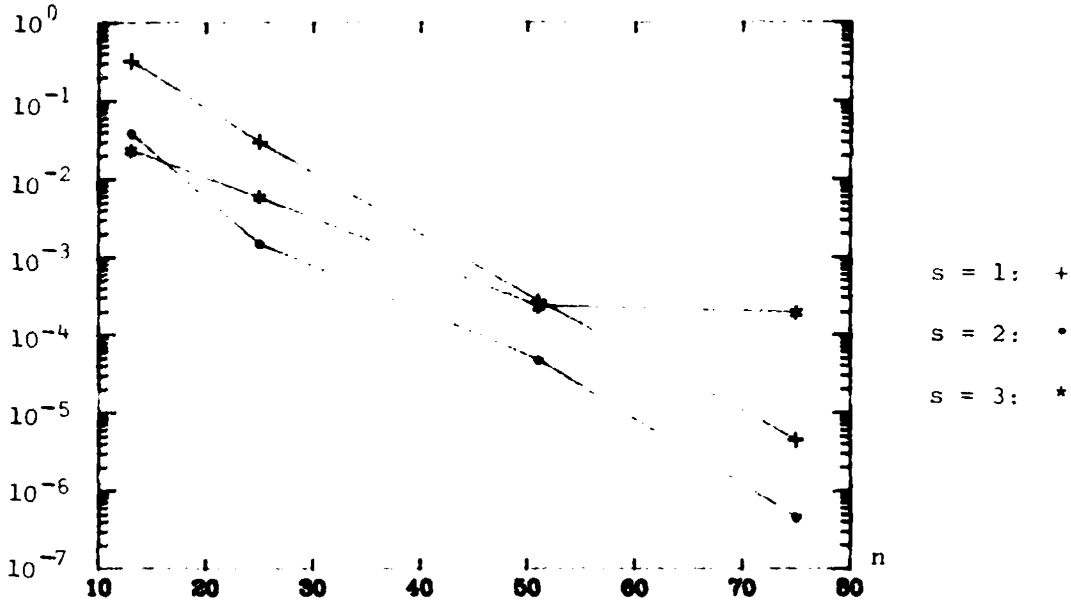

Figure 8.1

Only points with markers in figure 6.1 correspond to computed approximation errors. ■

Example 8.1 shows that the selection of s is of some importance, c.f. also example 3.2. When it is difficult to determine an appropriate allocation of poles, a crude determination of poles combined with discrete least squares approximation at the nodes $z_{k,m}$ can be a good strategy. For a justification when the poles all are at ∞, see [8]. Approximation by rationals with a different but fixed distribution of poles can be treated similarly as in [8]. Given m nodes $z_{k,m}$ and a distribution of poles $w_{k,n}$, one generally does not know a priori how to select the ratio m/n. One has to select several values of n and select the best of the computed approximants, see [8]. The difficulties in choosing m/n are illustrated in the next example.

Ex. 8.2. Consider the same approximation problem as in example 8.1 with $s = 2$, but let $\frac{n}{m} = 0.9$. For $m = 25, 51, 25,$ we let n be the even

integer closest to 0.9m. The "+" in figure 8.2 show the approximation error. The dots correspond to interpolation $n = m-1$ and are the same as in figure 8.1.

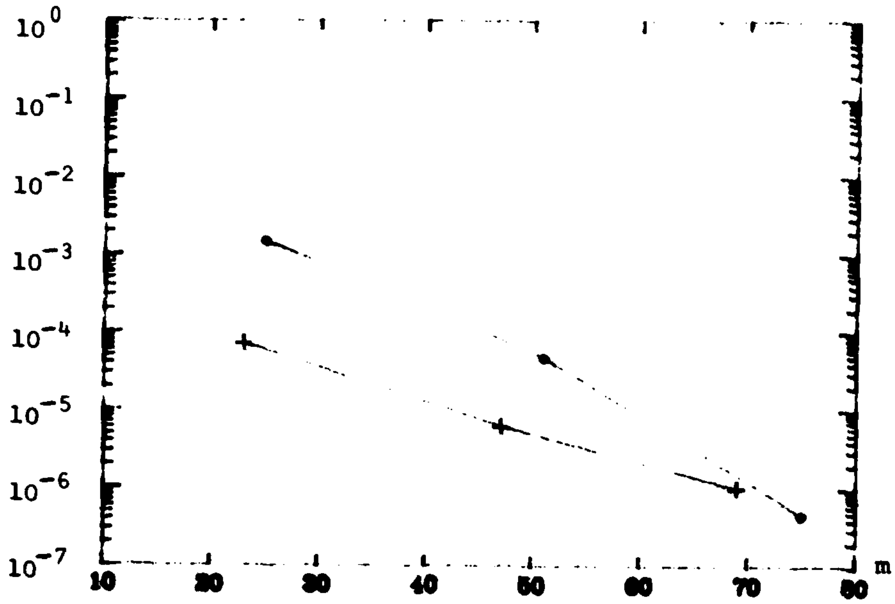

Figure 8.2

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REFERENCES

1. Curtiss, J. M., Convergence of complex Lagrange interpolation on the locus of the interpolation points, Duke Math. J., 32, 187-204, 1965.
2. Dahlquist, G. and Bjorck, A., Numerical methods, Prentice Hall, Englewood Cliffs, NJ, 1974.
3. Gaier, D., Vorlesungen uber Approximation im Komplexen, Birkhauser, Basel 1980.
4. Gautschi, W., Questions of numerical condition related to polynomials, in Recent advances in numerical analysis, eds. de Boor, C. and Golub, G. H., Academic Press, New York 1978.
5. Meiss, Th. and Markowitz, U., Numerische Behandlung partieller Differentialgleichungen, Springer, Berlin, 1978.
6. Reichel, L., A numerical method for analytic continuation of conformal mappings, Report TRITA-NA-8118, Dept. of Comp. Sci., Royal Institute of Technology, Stockholm, Sweden, 1981.
7. Reichel, L., Some computational aspects on rational approximation in the complex plane, Report TRITA-NA-8111, Dept. of Comp. Sci., Royal Institute of Technology, Stockholm, Sweden, 1981.
8. Reichel, L., On polynomial approximation in the uniform norm by the discrete least squares method, MRC Technical Summary Report #2472, Mathematics Reseach Center, University of Wisconsin-Madison, 1982.
9. Smirnov, V. I., and Lebedev, N. A., Functions of a complex variable, Iliffe Books, London, 1968.
10. Walsh, J. L., Interpolation and approximation by rational functions in the complex domain, American Mathematical Society, Providence, R.I., 1935.

11. Walsh. J. L., The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, Bulletin of the Amer. Math. Soc., 35, 499-544, 1929.

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Let Ω be a finitely connected closed point set in the complex plane with a piecewise smooth boundary $\partial\Omega$. The approximation of functions analytic on Ω by rational functions determined by interpolation or least squares approximation at preselected nodes is discussed. Attention is focussed on simple methods for selecting an appropriate rational space and obtaining a fairly well-conditioned rational basis. Applications include the determination of conformal mappings. Numerical examples illustrate the approximation method.

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